Overcoming Financial Frictions with the Friedman Rule

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Abstract

A general equilibrium model with financial frictions in which individuals may encounter unobservable investment opportunities is developed along the lines of Kiyotaki and Moore (2012). I study efficiency properties induced by money and monetary policy when financial frictions prevent optimal equilibrium allocations. By providing closed-form solutions to all prices, allocations, welfare and especially the distribution of individuals with respect to assets, I show that the Friedman rule achieves maximal social welfare, independent of how tight the financial constraints may be. The same level of welfare would be induced by an omniscient central planner able to verify who has an investment opportunity.

Keywords: insurance, efficiency, financial frictions, monetary policy, asset markets, heterogeneous agents.

JEL Classification: E52, E44.

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1 Introduction

I analyze the efficiency properties of money in a model in which money is essential. Individuals who are buffeted by idiosyncratic shocks of investment opportunities – the possibility of creating value in the form of new units of capital – resort to financial assets to fund the investment. One of these assets is equity, which they can issue against the capital they create. The other asset is money, which is provided by the monetary authority. Because no special function is assumed for money, it is valued endogenously in the economy, rendering it essential.

The model belongs to a class of general equilibrium models developed in Kiyotaki and Moore (2005) and Kiyotaki and Moore (2012) and further analyzed by Del Negro et al. (2011), Bigio (2012) and Shi (2015). I modify the environment of these papers, impose linearity of preferences as in Taub (1988), and assume complete depreciation. The resulting environment permits the model to be solved entirely in closed form. This represents one of the main contributions of the paper, as all equilibrium objects are analytically found: prices, allocations, welfare and especially the distribution of individuals with respect to assets. I use the model to analyze the efficiency properties and welfare consequences of money and monetary policy in a stationary environment.[1]

Under certain conditions, agents in the model would value insurance, in the sense that individuals without an investment opportunity would be willing to pay to obtain funds when an opportunity arrives. The model reveals divergence in the intertemporal marginal rate of transformation among agents with and without investment opportunities, which creates room for insurance that is valued by agents despite their linear preferences. An efficient insurance institution would indeed transfer resources from those individuals who do not have an investment opportunity to those who do. To accomplish this, insurance companies would need to elicit accurate information on the existence of an investment opportunity. The fact that we do not observe this type of insurance in reality is indicative of how costly this activity would be; hence, I assume that insurance is unfeasible and justify this assumption by requiring that the availability of the investment opportunity be unobservable. What other economic mechanisms can be useful in this setting? Equity might be one instrument used to accomplish the task, yet a moral hazard argument in the vein of Kiyotaki and Moore (2005) exogenously prevents agents with an investment opportunity from raising the full value of capital created. Would money offer an improvement?

[1]All of the models developed in the cited papers use the framework of Kiyotaki and Moore (2005) (θ, ϕ), θ being the maximum amount of claims on new capital that can be sold per period and ϕ the maximum amount of claims on existing units of capital. Most of these papers focus on the consequences of fluctuations in ϕ, representing "liquidity shocks". In this paper, complete depreciation is assumed, and hence, this liquidity friction is not taken into account and θ becomes preponderant. I present a fuller description of the contribution of the present paper with respect to some of these references subsequently in this introduction and in subsection 2.1.1 in section 2.

This paper is concerned with this issue. Value is created by those individuals who find investment projects; these individuals–entrepreneurs–finance capital creation, partially by issuing claims that are purchased by the other type of individuals–lenders. When money is valued, entrepreneurs may also use money to purchase goods from lenders to feed the capital production technology. There is no imposed requirement that goods be purchased with money; entrepreneurs find that it is in their own interest, under certain conditions, to use money to raise real funds. Lenders may also find it in their own interest to sell goods in exchange for money. They increase their stock of money in anticipation of the arrival of an investment opportunity, thereby allowing them to have greater financial resources for finance capital creation. It can then be deduced that money demand in this model is precautionary.

With equity being transacted and money circulating in the economy, one might think that efficiency can be achieved because equity allows for transferring resources from lenders to entrepreneurs and money allows for insurance against opportunities for investment projects. I show that while efficiency is improved, the same welfare achieved under an efficient insurance arrangement is not achieved. When entrepreneurs repeatedly find investment opportunities, they eventually run out of money, and because they can only sell claims up to a certain fraction of capital, the economy does not expand sufficiently to attain what perfect insurance would deliver: optimal allocations and the elimination of the divergence of marginal rates of transformation.

Would the monetary authority increasing the amount of money in the economy improve welfare? The stationary environment studied admits perpetual increases in the stock of money proportional to the previous period's stock. Money is injected as a "helicopter drop" into the economy, with each agent (regardless of whether the agent is an entrepreneur or lender) receiving the same amount. After this, individuals will interact in the market, transacting their money holdings. Entrepreneurs sell all of their money holdings in pursuit of goods for capital creation but with the expectation of higher amounts of nominal money in the future; under flexible prices, inflation is projected to be positive such that money will decrease in value. Lenders' demand for money would fall while claims on capital rise, causing a reduction in the price of money and in the funds entrepreneurs obtain for financing capital. Therefore, money is not superneutral: Anticipated inflation reduces the value of transacted money, which is the asset that enables the transfer of goods toward the production of investment.[2]

If inflation is detrimental, would deflation realize the same welfare as successful insurance? Friedman (1969) states that the money quantity rule of deflating at the internal rate of time preference attains optimality in some

[2]This contrasts with models that assume money to have a special property or a specific role, such as Sidrawski (1967) or Cooley and Hansen (1989), where the non-neutrality result arises due to consumption-leisure substitution under inflation.

settings. Indeed, this is the case in this environment. Deflation is beneficial because it increases the return on money, causing lenders to demand more of it and its price to increase sufficiently such that entrepreneurs end up with higher real balances to finance capital. The Friedman rule, by equating the return on money with the return on the real asset and the discount rate, eliminates the opportunity cost of holding money and successfully equates marginal rates of transformation among individuals, delivering the same welfare as an economy operating under a perfect insurance scheme.

Methodologically, this paper belongs to the tradition of models in which heterogeneity is central, such as Lucas (1980), Diamond and Dybvig (1983), Curdia and Woodford (2009), or Wen (2015). The specific heterogeneity present in this paper is akin to Carlstrom and Fuerst (1997), Bernanke and Gertler (1999), Fiore and Tristani (2007), and Kiyotaki and Moore (2012). In that money and credit or equity may coexist, this paper is also akin to models in the "search" tradition such as Aiyagari, Wallace and Wright (1996), Mills (2007), Telyukova and Wright (2008), Nosal and Rocheteau (2013), and Telyukova and Visschers (2013).

This paper is most similar to Kiyotaki and Moore (2012), but relevant modifications are introduced. They assume two different groups of agents that they call entrepreneurs and workers. Entrepreneurs may face investment opportunities, produce output and hire workers. I argue, in Section 2 of the paper, that such an environment is not conducive to proper closed-form solutions for policy functions or the existence of equilibrium distributions. I assume instead that all agents may find investment projects, and hence, there is no exogenous separation between entrepreneurs and workers and the economy's output is produced by a CRS firm that rents both capital and labor. With these features, the model developed is similar to the Neoclassical Growth Model. However, I further assume that preferences are linear in consumption, as in Taub (1988) and Taub (1994), which facilitates finding closed-form solutions for both policies and distributions of individuals with respect to assets. I also employ the assumption of full depreciation, which allows enormous simplifications in the algebraic computations of the solutions.

In terms of focus and questions pursued, the model presented in this paper differs from both Taub (1988) and Kiyotaki and Moore (2012). Kiyotaki and Moore (2012) study the effects of government purchases of assets backed by private capital, when these assets are illiquid in the economy. I focus on the welfare properties of money and the optimality of the Friedman rule in a stationary environment and do not consider a role for liquid assets due to the assumption of full depreciation of capital.[3] A number of papers, such as Taub (1988),

[3]While related papers have suggested the optimality of the Friedman rule with the type of financial frictions used in this paper,

Telyukova and Visschers (2013) and Wen (2015), study environments in which money has precautionary roles, as in the current paper. There are important differences, however. These papers study how money can be demanded for precautionary reasons against demands to consume, shocks that directly affect individuals' utility. I consider idiosyncratic shocks in the form of investment opportunities: money is demanded to finance investment opportunities when they arrive. This role of money is also taken from Kiyotaki and Moore (2012), but they do not study its welfare properties or the optimality of the Friedman rule.

Idiosyncratic uncertainty and heterogeneity are important to understand the demand for money on a theoretical and empirical level, and several constructs have been employed to address aggregation. In this respect, the paper is related to the contributions in the "search" tradition such as Nosal and Rocheteau (2013) and Telyukova and Visschers (2013). These papers approach aggregation by imposing quasi-linear preferences and division into centralized and decentralized subperiods. In these constructs, the influence of heterogeneity over many periods is muted because distributions are reset every second subperiod.[4] By contrast, in the present paper, the effects of idiosyncratic uncertainty carry over potentially infinite periods.

The paper is organized as follows: Section 2 presents the model, while Section 3 characterizes the economy to support the exposition by abstracting from money. To compare the model's results with money in terms of welfare, Section 4 introduces insurance, assuming that the availability of investment opportunities is observable. Section 5 analyzes the complete model with money, and Section 6 discusses the Friedman rule. Section 7 concludes.

2 The Model

2.1 Environment

The economy is populated by a measure one of infinitely lived individuals who seek to maximize:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s c_{t+s}, \quad 0 < \beta < 1. \tag{2.1}$$

for example, Kocherlakota (2005), I formally demonstrate the result.

[4] I would like to acknowledge an anonymous referee for bringing to my attention several papers in the search tradition and how the authors address heterogeneity.

The linearity of preferences, as in Taub (1988) and Taub (1994), is convenient for obtaining analytical results throughout the paper and highlights that financial transactions and precautionary demand for money can arise without risk aversion. The expectation operator $\mathbb{E}_t$ refers to an uninsurable idiosyncratic risk. Each period, with probability π, an agent has an "investment opportunity" by which he can transform units of the consumption good into units of capital.[5] All agents are endowed with one unit of labor, and hence, any capital in their hands along with labor is rented in each period to a CRS firm.

The status of the individuals is denoted by z, with $z = 1$ for an agent who has an investment opportunity (this agent will also be called an "entrepreneur") and $z = 0$ for an agent without an investment opportunity (who will also be referred to as a "lender"). Lender is an appropriate name for the latter because, as we will see, in equilibrium, lenders will partially finance the capital creation of entrepreneurs.

There are two financial assets in this economy, claims on capital (denoted n) and money (m denoting *real balances*). As entrepreneurs may issue claims on future capital in any period, lenders may save by purchasing these claims, or they may also use money. Let $v(n, m; z)$ be the value function for an agent with states (n, m) and status $z \in \{0, 1\}$.[6] For a current lender, the Bellman equation is:

$$v(n, m; 0) = \max_{c,n',m'} \left[c + \beta\pi v(n', m'; 1) + \beta(1 - \pi)v(n', m'; 0)\right], \tag{2.2}$$

subject to:

$$c + qn' + \gamma m' \leq w + rn + m + \tau \tag{2.3a}$$

$$n' \geq 0, \quad c \geq 0, \quad m' \geq 0, \tag{2.3b}$$

where w and r are the real rental rates of labor and capital, respectively; hence, factor income $w + rn$ is supplemented with money $m = \mu u$, where u is nominal money and μ is the price of money, and transfers of money are $\tau = \mu T$.[7] $T = (\gamma - 1)M^s$ are nominal transfers by the monetary authority, where γ is the gross rate of money growth and M^s is the nominal stock supplied to the economy. Income is used for consumption, purchases of claims at price q and purchases of real balances at price γ.[8] Restrictions in (2.3b) show that purchases of

[5]This type of heterogeneity and capital production technology has a substantial tradition in the financial literature in macroeconomics. Versions of this type of heterogeneity have been used by, among others, Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), Bernanke and Gertler (1999), Kiyotaki and Moore (2012) and Salas (2013).

[6]To simplify notation, I avoid using a subindex for an individual's objects such as n_i and m_i and instead simply use lower-case letters; to denote aggregate variables, I use upper-case letters.

[7]μ corresponding to the inverse of the price level; the price level is not used because money may not be valued.

[8]In this stationary environment, real balances are constant over time, and hence, it must be the case that $\mu' M' = \mu M$, where M is aggregate nominal balances. As $M'/M = \gamma$, it follows that $\gamma = \mu/\mu'$, which equals the gross inflation rate. Hence, if an agent

claims, money and consumption must be positive. Note that the model will be solved in a stationary state under the assumption that the economy has settled on constant prices (except μ, which may vary according to proportional variations in the stock of money). Money may not have any value, as it is not required to accomplish any specific function. As in the framework of Kiyotaki and Moore (2012), money will be valued endogenously in this economy.

Current entrepreneurs, by assumption, are allowed to create capital on a one-to-one basis with the consumption good as input. Their problem can be expressed as:

$$v(n, m; 1) = \max_{c, n', m', k'} \left[c + \beta\pi v(n', m'; 1) + \beta(1-\pi)v(n', m'; 0) \right], \tag{2.4}$$

subject to:

$$c + k' + qn' + \gamma m' \leq w + rn + qk' + m + \tau \tag{2.5a}$$

$$n' \geq (1-\theta)k', \quad c \geq 0, \quad m' \geq 0. \tag{2.5b}$$

Unlike the lenders' constraint, on the left-hand side, we have k', which is the cost of capital creation, and on the right–hand side, qk' is the income from selling claims. $\theta \in (0, 1)$ is a measure of financial frictions, an ad hoc restriction on claims on capital. The first inequality in (2.5b) states that the entrepreneur can sell at most θ of k'. Hence, capital cannot be completely self-financed. An entrepreneur must claim at least $1-\theta$ of the capital he creates for himself. This is taken exactly as in Kiyotaki and Moore (2005) and Kiyotaki and Moore (2012). The justification is moral hazard. Entrepreneurs act as managers of capital because by selling equity in the current period, an entrepreneur promises that the rental income of the capital he creates and on which claims are sold will be given to the actual purchaser of the claims. Entrepreneurs might not deliver on their promise, and thus, they are constrained to selling only up to $\theta k'$ of claims; therefore, they cannot self-claim below $(1-\theta)k'$.[9]

Note that full depreciation is assumed, and hence, rn is the only income from claims on capital chosen in the previous period. Comparing this to the Kiyotaki and Moore (2012) environment, they maintain the assumption of partial depreciation, which is relevant for them because they study the liquidity properties of existing claims on un-depreciated capital. They also introduce another friction, denoted ϕ, on transactions on existing claims on capital. In essence, agents can, within a given period, sell at most ϕ of claims on *existing* capital. By imposing

wishes to hold m' *real* balances for next period, he needs to purchase $\gamma m'$ units in the current period.

[9]Bigio (2012) studies more formally the informational problem that leads to this type of friction. Nosal and Rocheteau (2013), in a monetary matching model, also use this type of exogenous friction to account for many empirical facts concerning monetary policy and asset prices.

full depreciation, I disregard the liquidity properties of claims altogether; in subsection 2.1.1 below, I discuss further the implications of considering full depreciation and how it relates to the Kiyotaki and Moore (2012) environment.

Firms' optimization problem is standard and simple. CRS firms rent capital, produced by current entrepreneurs, and labor services, provided by all agents, in each period to maximize $[F(K,L) - rK - wL]$, with optimality conditions:

$$r = F_K(K^d, L^d), \quad w = F_L(K^d, L^d). \tag{2.5c}$$

The superscript denotes the demand for factors. Throughout the analysis, I will use the Cobb-Douglas production function: $Y = K^\alpha L^{1-\alpha}, \quad 0 < \alpha < 1$.

2.1.1 Comparison with Kiyotaki and Moore (2012) and discussion

Here, I discuss the main differences between the environment considered in this paper and that of Kiyotaki and Moore (2012). Kiyotaki and Moore (2012) consider two separate sets of agents: entrepreneurs and workers. Entrepreneurs with log utility are responsible for producing the consumption good in the economy, and they may have investment opportunities in a similar fashion as in the present paper.[10] Workers with GHH preferences work for the firms operated by entrepreneurs.[11] That setup delivers budget sets for their entrepreneurs similar to (2.3a) and (2.5a) above but where profits from producing output enter as income for entrepreneurs instead of w. Another important difference is that they consider partial depreciation of capital. This is important in their model, as they seek to study illiquid claims on existing capital. In this case, the financial constraints are modified with respect to those in (2.3b) and (2.5b). They consider illiquid claims on existing capital in addition to the financial constraint reflected in θ. They assume that the speed at which claims on existing capital can be sold is exogenous and given by another parameter ϕ. Hence, the finance constraints they employ are given by:[12]

$$n' \geq (1-\theta)x + (1-\phi)(1-\delta)n, \quad n' \geq (1-\phi)(1-\delta)n, \tag{2.6}$$

[10] Hence, what Kiyotaki and Moore (2012) call entrepreneurs are not equivalent to entrepreneurs in the present paper. I define entrepreneurs as agents that in addition to obtaining labor income are able to produce capital in the economy. All agents supply factor services to CRS firms that produce the consumption good in the economy.

[11] GHH preferences refer to Greenwood, Hercowitz and Huffman preferences (Greenwood, Hercowitz and Huffman, 1988). These preferences exhibit an absence of the wealth effect in labor, thus partially explaining why workers in Kiyotaki and Moore (2012) do not participate in the asset markets.

[12] This is part of the (θ, ϕ) framework alluded to in footnote 1 in Section 1.

for agents with and without the investment opportunity, respectively. x is investment, and δ is the depreciation rate.[13] As it will also occur in this paper, as we will see shortly, under certain conditions, agents with investment opportunities saturate their finance constraint and end up with a non-negativity constraint on investment as a relevant constraint, while agents without investment opportunities still face the second constraint in (2.6). This lengthy discussion is necessary to portray how the modifications in my paper overcome certain problems in the Kiyotaki and Moore (2012) construct. As explained above, profits from producing the consumption good enter their entrepreneurs' budget constraint. After some manipulations of this profit function, they are able to express all income in the entrepreneurs' budget as linear functions of the assets.[14] As entrepreneurs have log utility, if they could freely choose next period assets without facing state-dependent constraints in (2.6), then a result first derived in Samuelson (1969) could be used to find closed-form solutions for policy functions: in essence, a simple solution arises for agents in that they end up consuming $(1-\beta)$ of all income and saving the rest. Hence, the policy functions used by Kiyotaki and Moore (2012) are not correct, unless one assumes that none of the entrepreneurs without investment projects and none of the entrepreneurs with investment projects saturate their finance and non-negativity constraints on investment, respectively. As I do not follow the Kiyotaki and Moore (2012) environment, I am not confronted with this problem. I am able to find closed-form solutions for policies by assuming linearity of preferences.

With linear preferences and partial depreciation, closed-form solutions for policies and the distribution would not be possible to obtain in my model unless it is assumed that $\phi = 1$, that is, existing claims on capital are fully liquid. The reason for this result is explained in greater detail in subsection 3.2. Thus, many instances of my model could be solved with partial depreciation assuming full liquidity of existing capital, but the resulting equations become algebraically too complex, and the objects too non-linear, to the extent that welfare comparisons are not possible to undertake analytically. Nothing substantial seems to be lost by assuming full depreciation, and hence, throughout the paper, I use this simplifying assumption.

2.2 Definition of equilibrium

Definition *A stationary recursive competitive equilibrium consists of the following factor prices (r, w): price*

[13] Of course, finance constraints in (2.3b) and (2.5b) are special cases of (2.6) when $\delta = 1$ and $x = k'$.

[14] It is important that there are no other terms in those budget constraints that are not multiplying some asset, either money or claims on capital. Thus, for example, I am unable to use their approach to solve for closed-form solutions to policy functions in my model because although there is linearity in assets in (2.3) and (2.5), the wage rate appears in both without multiplying any assets.

of capital q, price of money μ; policy functions: for consumption $c(n,m;z)$, next period claims $g(n,m;z)$, next period money $h(n,m;z)$, capital $k'(n,m)$, probability measures $\Psi(n,m;z)$, total aggregate capital K and aggregate real balances H, such that:[15]

1. *$c(n,m;z)$, $g(n,m;z)$, $h(n,m;z)$ and $k'(n,m)$ maximize an individual's utility subject to the constraints*
2. *at given r, w firms maximize profits*
3. *the claims on the capital market clear:*

$$K^s = \sum_z \int n d\Psi(n,m;z) \tag{2.7a}$$

4. *the capital and labor markets clear:*

$$\begin{aligned} K^d &= K^s = K && (2.7b)\\ L^d &= \sum_z \int d\Psi(n,m;z) = L^s = 1 = L && (2.7c) \end{aligned}$$

5. *investment demand equals savings:*

$$\sum_z \int n'(n,m;z) d\Psi(n,m;z) \quad = \quad \int k'(n,m) d\Psi(n,m;1) \tag{2.7d}$$

6. *the money market clears:*

$$\sum_z \int m d\Psi(n,m;z) \equiv \mu M^s = H \tag{2.7e}$$

7. *the probability distribution is time invariant:*

$$\Psi(\tilde{n},\tilde{m}) = \pi \int_{\mathcal{B}(\tilde{n},\tilde{m};1)} d\Psi(n,m) + (1-\pi) \int_{\mathcal{B}(\tilde{n},\tilde{m};0)} d\Psi(n,m) \tag{2.7f}$$

where:

$$\mathcal{B}(\tilde{n},\tilde{m};z) = \{(n,m) : n \geq 0, m \geq 0, g(n,m;z) \leq \tilde{n}, h(n,m;z) \leq \tilde{m}\} \tag{2.7g}$$

[15]The i.i.d. assumption implies that $\Psi(n,m;1) = \pi\Psi(n,m)$ and $\Psi(n,m;0) = (1-\pi)\Psi(n,m)$, where $\Psi(n,m)$ is the distribution of the whole population with respect to assets.

3 The economy without money

I begin with a version of the environment without money; it is a useful benchmark for discussing the efficiency properties of the model.

3.1 Solving the Model: A guess-and-verify strategy

Regarding the entrepreneur's constraint (2.5a), it is evident that whether q is higher or lower than one is important for his decision of how much capital to create. I guess that:

$$q > 1, \tag{3.1}$$

determine the agent's decisions, and identify a condition such that the *equilibrium* value of q satisfies (3.1).

When (3.1) holds, the income from creating capital is higher than its cost, and for a given n' in (2.5a), entrepreneurs seek to invest as much as possible, saturating their financial constraint in (2.5b).[16] As this financial constraint binds, $n' = (1-\theta)k'$, it is possible to substitute out k' from (2.5a), resulting in the feasibility set:[17]

$$c + q^e n' \leq w + rn, \quad n' \geq 0, \quad c \geq 0, \quad \text{where} \quad q^e \equiv (1-q\theta)/(1-\theta). \tag{3.2}$$

q^e is the *effective* price of equity for entrepreneurs. A fraction θ of the capital created is financed by selling claims in the market; therefore, they pay only $1-q\theta$ of a unit of capital with their own funds. The remaining $1-\theta$ of that unit is self-claimed as equity, and the effective price of a *unit* of equity is then $q^e = (1-q\theta)/(1-\theta)$. Under assumption (3.1):

$$q^e < 1 < q, \tag{3.3}$$

and hence, the cost of transforming current consumption into future consumption is lower for current entrepreneurs than for lenders. Because individual status changes randomly, agents face heterogeneous intertemporal "marginal rates of transformation". At those prices, a current lender has to sacrifice q units of consumption,

[16]When $q = 1$, they would be indifferent on how much to invest, and $q < 1$ can be excluded as an equilibrium outcome because in this case, investment would be zero. The case in which $q = 1$ will be examined below.

[17]$n' > 0$ in (3.2) is the non-negativity constraint on investment, as $n' = (1-\theta)k'$ holds when $q > 1$.

whereas a current entrepreneur only q^e. In comparing this cost with the benefit, individuals need to compute the expected marginal value of equity, which can be computed from (2.2) and (2.4) once these value functions are identified. I use a guess-and-verify strategy for this task and assume:

$$v(n;z) = A_z + B_z n, \quad z = \{0,1\}, \tag{3.4}$$

where A_z and B_z are undetermined coefficients. Proposition 1 presents the value and policy functions found.[18]

Proposition 1. *Under* (3.1)*, the value and policy functions for individuals are:*

$$v(n;0) = \left\{\beta\pi\frac{q}{q^e} + 1 - \beta\pi\right\}\frac{w}{1-\beta} + rn \tag{3.5a}$$

$$v(n;1) = \left\{[1-\beta(1-\pi)]\frac{q}{q^e} + \beta(1-\pi)\right\}\frac{w}{1-\beta} + \frac{q}{q^e}rn \tag{3.5b}$$

$$g(n;0) \in \left[0, \frac{w+rn}{q}\right], \quad c(n;0) = w + rn - qg(n;0) \tag{3.6a}$$

$$c(n;1) = 0, \quad (1-\theta q)k'(n) = w + rn, \quad g(n;1) = (1-\theta)k'(n). \tag{3.6b}$$

Is easy to show that the value functions are increasing in w, r and q/q^e.[19] It can also be shown that $v(n;1) > v(n;0)$, reflecting the advantage of creating capital that current entrepreneurs enjoy. Once equilibrium prices are identified, (3.5) will be used to perform welfare comparisons. These comparisons are also useful for determining the benefit of acquiring claims, while we know that the agents differ in the cost of purchasing them. The explanation of Proposition 1 in the Appendix establishes that the following relationships must hold in equilibrium:

$$q^e < q = \beta\left[\pi\frac{q}{q^e}r + (1-\pi)r\right]. \tag{3.7}$$

Both entrepreneurs and lenders compare the cost with the benefit of acquiring a unit of equity. The benefit can be computed with the derivative of the value functions in (3.5). While for entrepreneurs, the benefit exceeds the cost, for lenders, benefits and costs are equal. This means that lenders are indifferent between consuming and saving, as is expressed in (3.6a). Entrepreneurs adopt a corner solution with zero consumption. The middle

[18] Proofs of most propositions are presented in the Appendix.

[19] While the positive effect on welfare of w and r is straightforward to understand, the positive influence of q/q^e is less obvious. A higher q (and hence lower q^e) is favorable for entrepreneurs because the down payment on investment $(1-\theta q)$ is decreased. This ratio also appears in $v(n;0)$ because a current lender expects to become an entrepreneur in the future; note that the ratio enters the value function multiplied by π.

equation in (3.6b) shows that the down payment to finance capital $(1-\theta q)k'(n)$ is financed with all factor income $w+rn$. The fraction of capital on which the entrepreneur is unable to issue claims is "self-claimed"; this is expressed in the third equation in (3.6b). The equality in (3.7) needs to hold because entrepreneurs are selling claims, and hence, lenders must not be at a corner purchasing zero claims for markets to clear, nor can they be at a corner *only* purchasing claims because in this case, consumption would always be zero for all agents.

The marginal benefit in (3.7) is composed of the discounted expected gain. With probability $1-\pi$, carrying a unit of equity would deliver r units for consumption; with probability π, the agent is an entrepreneur and also gaining r units. However, because of the corner solution, these units are not consumed but valued at price $q/q^e > 1$, which shows that asset prices favor entrepreneurs.[20]

Expected returns and their relationship with the discount factor are easily obtained from (3.7) above and made explicit here for future reference:

$$\mathcal{R}_0 \equiv \pi\frac{r}{q^e} + (1-\pi)\frac{r}{q} = \frac{1}{\beta} < \pi\frac{r}{q^e}\frac{q}{q^e} + (1-\pi)\frac{r}{q^e} \equiv \mathcal{R}_1, \tag{3.8}$$

an inequality that holds when $q > 1$, where $\mathcal{R}_z$ is the expected return to equity for an agent with status z.

Society's welfare is found by aggregating individuals' values with individuals distributed according to their asset holdings:[21]

$$V = (1-\pi)\int v(n;0)d\Psi(n) + \pi\int v(n;1)d\Psi(n), \tag{3.9}$$

with values defined in (3.5). For (3.9) to be well defined, we need to show that $\Psi(n)$ exists and, given the linearity of welfare on n, that the first moment $\int nd\Psi(n) = K$ is well defined. Indeed the existence of equilibrium itself also requires these objects to exist. This is tackled in the next subsection.

[20] Intuitively, the r units of goods that an entrepreneur obtains can be "transformed" at rate $1/q^e$ into equity using the investment technology. Then, r/q^e are units of claims that are valued at market price q to form the gain in the first term in the brackets in (3.7).

[21] Once equilibrium is found, it is possible to identify a closed-form solution to welfare. This is described in equation (A.7) in the Appendix, in the proof of Proposition 4.

3.2 Existence of Equilibrium

As lenders face prices such that the cost of equity exactly matches the discounted expected gain, at the individual level, the lenders' actions are not specified. However, we know that a measure π of current entrepreneurs is selling claims to lenders; therefore, in equilibrium, a sufficiently large measure of lenders must be buying non-zero amounts of claims. Assumption 1 below requires all lenders to have the same policy function and, moreover, that each purchase the same amount of equity for the next period ζ, which, in equilibrium, must be greater than zero.

Assumption 1. Homogeneity of lenders.

$$g(n;0) = \zeta. \tag{3.10}$$

Assumption 1 enables the selection of an equilibrium. Different equilibria may arise if we allow for heterogeneity among lenders in their equity holdings, yet, as the value function $v(n;0)$ in (3.5) was derived with the policy (3.6a), any other equilibria will deliver the same individual and aggregate welfare.[22]

ζ will be determined *endogenously* in such a way that all lenders exactly acquire the aggregate fraction of claims on capital issued by entrepreneurs in any given period, being strictly positive. To show this, we require the existence of aggregate values, which will be verified shortly.

Assumption 1 is important to find closed-form solutions for the distribution, as mentioned in subsection 2.1.1, and is related to modifications to the environment in Kiyotaki and Moore (2012). Note that this assumption does not violate the restriction $n' \geq 0$ in (2.3b) for lenders. If partial depreciation were allowed along with illiquid claims on existing capital, this constraint would be $n' \geq (1-\phi)(1-\delta)n$; hence, Assumption 1 cannot be imposed for all levels of equity. Assuming full liquidity of the asset, $\phi = 1$, Assumption 1 remains valid, and thus, existence of Ψ can be shown analytically even under partial depreciation. Notwithstanding, I opted to assume full depreciation because the algebraic solutions for prices and values are too complex and welfare comparisons are not possible to undertake analytically otherwise.

[22]Multiplicity of equilibria arise in the sense that different assumptions on $g(n;0)$ imply different equilibrium allocations at the individual level and different distributions $\Psi(n)$. However, if equilibria exist, then all of them must have the same first moment $\int n d\Psi(n) = K$. This is evident in the equilibrium conditions specified below in equation (3.13). This implies that equilibrium prices are independent of a specific assumption on $g(n;0)$ and so is welfare V, as (3.9) is linear in n.

Proposition 2. *Under Assumption 1,* $\Psi(n)$ *and its associated density are:*

$$\Psi\left(n_i\right)=1-\pi^i,\ d\Psi=1-\pi^i-\left(1-\pi^{i-1}\right)=(1-\pi)\pi^{i-1},\quad i=1,2,3,... \tag{3.11}$$

The support $\{n_i\}_{i=1}^{\infty}$ *is defined by:*

$$n_i=\zeta\left(\frac{r}{q^e}\right)^{i-1}+\frac{w}{q^e-r}\left[1-\left(\frac{r}{q^e}\right)^{i-1}\right],\quad i=1,2,3,... \tag{3.12}$$

The resulting structure for the distribution is special and reflects the corner solutions and the homogeneity in their policies for lenders. Using (3.6b), entrepreneurs' equity holdings are $g(n;1)=(w+rn)/q^e$. All income is used to purchase claims at the effective price q^e. As long as they maintain their status, they will continue using all factor income to accumulate assets; however, when they become lenders, they all hold $g(n;0)=\zeta$ for the next period. Eventually all will "start" with ζ of assets, and then, a discrete fashion of accumulation ensues.[23] This is the intuition behind the expression in (3.12). Whether the distribution is bounded or unbounded in its support depends on q^e and r, and hence, to characterize the distribution completely, the equilibrium of the economy needs to be found.

Finding the equilibrium requires the existence of the first moment of the distribution. I again use a guess-and-verify method and assume initially that the first moment exists. A system of equilibrium is formed by four equations. The first is the equality in (3.7), and the second equation is the aggregate of the policy function for capital, the middle equation in (2.5b). Note that this requires the existence of the first moment because $\int n d\Psi(n)=K$. The third and fourth equations represent the demand for factors (2.5c):

$$\underbrace{(1-q\theta)K}_{\text{down payment}}=\underbrace{(w+rK)\pi}_{\text{entrepreneurs' own funds}} \tag{3.13a}$$

$$\underbrace{\frac{1}{\beta}}_{\text{rate of time preference}}=\underbrace{\pi\frac{r}{q^e}+(1-\pi)\frac{r}{q}}_{\text{expected return on equity}} \tag{3.13b}$$

$$r=\alpha K^{\alpha-1},\quad w=(1-\alpha)K^{\alpha}. \tag{3.13c}$$

[23] Here, again, one can see that without full depreciation or full liquidity of the asset, if partial depreciation were allowed, it would not be possible for all agents to eventually "start" with ζ and hence there would be no possibility of finding a closed-form solution in this environment.

Forming four non-linear equations for the unknowns: r, w, q, K.

In Equation (3.13a), the entrepreneurs' entire factor income is used to finance capital creation because they do not consume. As there is full depreciation, each period K must be created in the stationary equilibrium. Entrepreneurs do not entirely finance the capital stock because a fraction θ of claims on capital is sold at price q; this is the down payment in the right-hand side of (3.13a).

System (3.13) is a nonlinear system of three equations in three unknowns with a closed-form solution:

$$q = \frac{1-\pi}{\pi\left(\frac{1}{\alpha\beta}-1\right)+\theta}, \quad r = \frac{\alpha\left(\frac{1}{\alpha\beta}-1+\theta\right)}{\pi\left(\frac{1}{\alpha\beta}-1\right)+\theta}, \quad K = \left[\frac{\pi\left(\frac{1}{\alpha\beta}-1\right)+\theta}{\frac{1}{\alpha\beta}-1+\theta}\right]^{\frac{1}{1-\alpha}}. \tag{3.14}$$

These analytical expressions will shortly be used to analyze the effects of the financial constraint θ and the number of entrepreneurs π. However, before doing so, let me complete the description of equilibrium by verifying the existence of the distribution and its first moment.

Proposition 3. *The support of $\Psi(n)$ is unbounded above and:*

$$\int_{n\in\mathcal{B}} n d\Psi(n) < +\infty.$$

It is straightforward to demonstrate, using the prices in (3.14), that $r/q^e > 1$, which means that irrespective of how large equity holdings are, an entrepreneur will always acquire more. With probability π, an entrepreneur will increase his equity holdings, and the density of agents will asymptotically vanish when equity holdings approach infinity.

All individuals hold assets above or equal to ζ. Individuals who become lenders and are holding assets in excess of ζ dissave, which is reflected in the structure in (3.12). Hence, while the remaining entrepreneurs accumulate ever more, those who become lenders counteract the divergent effect on capital caused by entrepreneurs' behavior and average capital remains bounded. The remaining lenders, however, remain in their position, holding exactly ζ units of claims. However, what is ζ? With the value of K in (3.14), it is easy to find this equilibrium value, using (2.7d):

$$(1-\pi)\zeta = \theta K, \tag{3.15}$$

which states that claims on the stock of capital sold by entrepreneurs are purchased by all lenders.

We finally reach the point at which a condition can be found such that the assumption in (3.1) is satisfied, an assumption used throughout. With the analytical q in (3.14), it is straightforward to find that this assumption is satisfied when:

$$\pi < \alpha\beta(1-\theta). \tag{3.16}$$

To understand this condition and to compare further results to be derived below, the next subsection presents a benchmark, the situation in which all agents are homogeneous. For now, let me complete this section with a brief discussion of comparative statics.

Comparative statics

A less financially constrained economy means that θ increases and entrepreneurs are able to sell more equity for any capital created. Another way to increase capital creation is to increase the extensive margin, to have more entrepreneurs, a higher π. In Inequalities (3.17), I show the effects of changing these parameters on equilibrium objects, results that can be obtained by direct differentiation.

$$\frac{\partial q}{\partial \theta} < 0, \quad \frac{\partial K}{\partial \theta} > 0, \quad \frac{\partial V}{\partial \theta} > 0, \quad \frac{\partial q}{\partial \pi} < 0, \quad \frac{\partial K}{\partial \pi} > 0, \quad \frac{\partial V}{\partial \pi} > 0. \tag{3.17}$$

Relaxing the constraints drives the price of equity closer to the fundamental value of 1, increases aggregate capital, and increases welfare. That a more financially constrained economy drives the price of the asset up appears to be a general property in models of financial frictions of this type; see, for example, Bigio (2012) and Shi (2015). Simply stated, the remaining liquid fraction of capital becomes more valuable as the financial constraint tightens.[24] Here, I find that a reduced π also increases q. The idea is the same; there is less capital creation due to a lower extensive margin, thus making liquid capital more valuable. Welfare is increased for a less-constrained economy, both in θ and in π. There are aggregate general equilibrium effects behind this result. In particular, the higher capital stock created means a higher wage for all individuals. However, there are also more subtle effects that will be discussed in the context of the following sections.

[24] $\theta = 0$ is a special case of the equilibrium found above. In such a situation, entrepreneurs unable to sell any claims would face an effective price of $q^e = 1$ but still have an advantage because $q > 1$. Lenders will hold zero assets ($\zeta = 0$) and consume all of their factor income. Those who become entrepreneurs, then, will not have any capital income, but because they also have labor income w, they are able to buy back claims in the market. I note that in Nosal and Rocheteau (2013), when $\theta = 0$, they find that q attains its fundamental value and allocations can be optimal; I do not obtain this result here.

Now, let me turn to a benchmark situation in which all agents are alike. This will be helpful for understanding condition (3.16) and for subsequent sections.

3.3 Benchmark: Homogeneous agent version

Here, I sketch a model in which a measure one of agents seeks to maximize (2.1) but all agents can invest; in this case, all can save without using the credit market. This is in fact a special case of the Neoclassical Growth Model in which agents have linear preferences; they can all invest, and there is full depreciation. The equilibrium can be solved as a Pareto problem; in the stationary state, $1 = \beta r^*$ must hold.[25] The corresponding optimal aggregate allocations are:

$$K^* = (\alpha\beta)^{\frac{1}{1-\alpha}}\ , \quad C^* = (1-\alpha\beta)\,(\alpha\beta)^{\frac{\alpha}{1-\alpha}}\ , \quad Y^* = (\alpha\beta)^{\frac{\alpha}{1-\alpha}}. \tag{3.18a}$$

For capital, consumption and output, respectively. Aggregate welfare from (2.1) can be easily computed:

$$V^* = \frac{(1-\alpha\beta)\,(\alpha\beta)^{\frac{\alpha}{1-\alpha}}}{1-\beta}. \tag{3.18b}$$

How do allocations and welfare in (3.14) and (3.9) compare to the Pareto optimal allocations and maximal welfare in (3.18a) and (3.18b), respectively? Moreover, how does the answer depend on Condition (3.16)? The next subsection explores these issues.

3.4 Efficiency properties of the equilibrium

The results of the previous section help to provide intuition for Condition (3.16). Dividing (3.16) by $\alpha\beta$ and multiplying by K^*:

$$\pi\frac{K^*}{\alpha\beta} = \pi(\alpha\beta)^{\frac{\alpha}{1-\alpha}} = \pi Y^* < (1-\theta)K^*, \tag{3.19}$$

where I also used (3.18a). Because output is produced with a CRS production function, factor payments exhaust aggregate output, a fraction π of which is in the entrepreneurs' hands. They need to finance a fraction $(1-\theta)$ of

[25] I denote by * equilibrium quantities in this economy that are Pareto optimal.

any capital they create. Hence, Condition (3.19) reveals that the optimal value of capital cannot be sustained by entrepreneurs' income; they need to reduce investment. Under Condition (3.16), it can be immediately verified that:

$$q > 1, \quad r > r^* = \frac{1}{\beta}, \quad K < K^*. \tag{3.20}$$

The fact that $q > 1$ in a constrained economy resembles the result in Nosal and Rocheteau (2013), who also find that an illiquid asset will have an equilibrium price above its fundamental value.[26] The rental rate being higher than r^* is simply a reflection of the reduced stock of capital relative to the Pareto value.

It is not obvious by considering aggregate allocations, such as the stock of capital, that all individuals would prefer to live in a "frictionless" world. For example, when (3.16) is satisfied, we know that (3.3) is satisfied, but then, the ratio q/q^e is higher than one, a ratio that positively influences value functions in (3.5). Intuitively, when agents have an investment opportunity, they enjoy an advantage because they face an effective price of capital accumulation that is below its fundamental value, $q^e < 1$. However, there are other general equilibrium effects in a constrained economy, such as a reduced wage rate, that make society worse off when Condition (3.16) is satisfied relative to the frictionless case. This is stated in Proposition 4.

Proposition 4. *Under Condition* (3.16),

$$V < V^*. \tag{3.21}$$

What would happen if Condition (3.16) were not satisfied? This would be the case, for example, in a relatively unconstrained economy where π or θ are high enough. In such a case, none of the derivations above are valid. I examine here a limiting case in which:

$$\pi = \alpha\beta(1 - \theta). \tag{3.22}$$

In this case, $q = 1$ as can be seen in (3.14), all agents face the same feasibility sets, and heterogeneity is immaterial. However, as this is a limiting case of the analysis above, entrepreneurs undertake investment with zero consumption. This is admissible because they are actually indifferent with respect to how much investment

[26]This fundamental value is one here because capital can be created from the consumption good on a one-to-one basis.

to undertake, and such a decision does not affect their welfare. The value functions coincide for both types of agents, which yields the following social welfare:[27]

$$V = \pi \int v^*(n,1) d\Psi(n) + (1-\pi) \int v^*(n,0) d\Psi(n) = \frac{(1-\alpha\beta)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}}{1-\beta} = V^*. \tag{3.23}$$

Hence, the same allocations and welfare are obtained as in a frictionless Pareto economy. Note, therefore, that it is not necessary that the economy be completely unconstrained ($\theta = \pi = 1$) to attain optimal results.

When (3.22) holds, then marginal rates of transformation will of course no longer differ across individuals. This admits the following interpretation for the parameter θ. An agent who finds an investment idea does not want to miss out on the opportunity because he has insufficient funds to invest in the project. There is room for insurance in the constrained economy. Equity itself allows for self-insurance to some extent. By acquiring more claims, a current lender can receive more capital income in the next period when an investment opportunity may arrive, but when Condition (3.16) holds, the financial constraint faced by current entrepreneurs prevents enough claims from being sold to lenders. The lower θ is, the worse equity serves for insurance purposes. In the event that Condition (3.22) is satisfied, a sufficiently high θ for a given π makes equity a sufficiently liquid asset for self-insurance such that optimal allocations are reached. In the next section, I formally study the properties of an economy in which insurance is possible and (3.16) is satisfied.

4 Insurance

A key aspect of the model's results thus far is that if an agent has an investment opportunity, he would like to have sufficient resources to take advantage of it. We can imagine a welfare-enhancing institution that provides insurance by selling contingent claims to lenders for the eventuality of becoming entrepreneurs. Insurance would be valuable because the marginal rates of transformation differ across individuals.[28] This would require insurance institutions to be able to identify the status of the agents.

[27] In this case, the state of the system is still discrete and follows

$$n_i = \zeta \left(\frac{1}{\beta}\right)^{i-1} - \frac{\beta(1-\alpha)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}}{1-\beta} \left[1 - \left(\frac{1}{\beta}\right)^{i-1}\right], \quad i = 1, 2, 3, ...$$

[28] Hence, in this model, although the marginal rate of substitution does not change with status, due to the linearity of preferences, the marginal rate of transformation does change. This creates room for insurance demand.

For simplicity, I use an approach with a central planner who intervenes and *observes* the status of each agent.[29] A scheme could be implemented as follows: Each lender surrenders χ^l to the central planner, and entrepreneurs each receive χ^e units. Feasibility sets are modified from above:

$$c + qn' \leq w + rn - \chi^l, \quad n' \geq 0, \quad c \geq 0, \quad z = 0, \tag{4.1a}$$

and:

$$c + q^e n' \leq w + rn + \chi^e, \quad n' \geq 0, \quad c \geq 0, \quad z = 1. \tag{4.1b}$$

Using a guess-and-verify strategy as before, assuming that the value of the parameters is such that $q > 1$, corner solutions for entrepreneurs and indifference for workers arise just as before. The following proposition shows the value functions conditional on status, which can be obtained in closed form (the proof is omitted because it parallels the previous case).

Proposition 5. *Under assumption $q > 1$, value functions for individuals are:*

$$v(n;0) = \left\{ \beta\pi \frac{q}{q^e} (w + \chi^e) + (1 - \beta\pi) \left(w - \chi^l \right) \right\} \frac{1}{1-\beta} + rn \tag{4.2a}$$

$$v(n;1) = \left\{ [1 - \beta(1-\pi)] \frac{q}{q^e} (w + \chi^e) + \beta(1-\pi) \left(w - \chi^l \right) \right\} \frac{1}{1-\beta} + \frac{q}{q^e} rn. \tag{4.2b}$$

We can see that the value functions in (3.5) are special cases of (4.2) when $\chi^l = \chi^e = 0$. Solving for the equilibrium prices and allocations requires solving a non-linear system that is exactly equal to (3.13), with the only difference being that the aggregate entrepreneurs' constraint (3.13a) is replaced with:

$$\underbrace{(1 - q\theta)K}_{\text{down payment}} = \underbrace{(w + rK + \chi^e)\pi}_{\text{entrepreneurs' own funds}}, \tag{4.3}$$

where it is evident that the additional resources relax the financial constraint that entrepreneurs face. The feasibility of the scheme implies that all resources collected from current lenders end up in entrepreneurs' hands. The system has a simple analytical solution, assuming that the fee collected from each lender is proportional to the aggregate capital income of the economy: $\chi^l = \chi r K$. Then, the amount of goods that each entrepreneur

[29] Equivalently, competitive private insurance institutions could be introduced that make zero profits by selling contingent claims to lenders for the eventuality of becoming entrepreneurs.

receives is:[30]

$$\chi^e = \frac{1-\pi}{\pi}\chi r K, \tag{4.4}$$

which states that all income collected from lenders $(1-\pi)\chi^\ell$ is equally divided among the π entrepreneurs. The resulting system (4.3), (3.13b) and (3.13c) has a closed-form solution; for completeness, I present the equilibrium values here:[31]

$$q^I = \frac{1-\pi}{\pi\left(\frac{1}{\alpha\beta}-1\right)+\theta+\frac{1-\pi}{\beta}\chi}, \quad r^I = \frac{\pi\left(\frac{1}{\alpha\beta}-1\right)+\frac{1-\pi}{\beta}\chi+\theta\pi}{\left[\frac{\pi}{\alpha}+(1-\pi)\chi\right]\left[\pi\left(\frac{1}{\alpha\beta}-1\right)+\theta+\frac{1-\pi}{\beta}\chi\right]}, \quad K^I = \left(\frac{\alpha}{r^I}\right)^{\frac{1}{1-\alpha}}. \tag{4.5}$$

Note that by increasing χ, χ^e in (4.3) increases, which tends to relax the entrepreneurs' financial constraint; this is obvious and anticipated given the discussion above regarding the benefits of providing entrepreneurs with more resources. However, there are intricate general equilibrium effects than can be appreciated in (4.5). Instead of focusing on the effects of insurance on allocations and prices, I simply focus on the effects on welfare. Because lenders are the ones who consume, it is not immediate that by surrendering χ^ℓ goods, welfare is improved for all agents. Welfare can be found in closed form again as a function of χ:

$$V^I(\chi) = (1-\pi)\int v(n;0)d\Psi(n)+\pi\int v(n;1)d\Psi(n) = \left[\frac{1-\alpha\beta}{\alpha(1-\beta)}\left(1-\pi+\pi\frac{q^I}{q^{eI}}\right)+\frac{(1-\pi)\chi}{1-\beta}\left(\frac{q^I}{q^{eI}}-1\right)\right]r^I K^I. \tag{4.6}$$

Optimal allocations with insurance require that χ maximizes society's welfare:

$$\chi^* = \arg\max V^I(\chi). \tag{4.7}$$

It can be shown that the optimal value solving (4.7) satisfies:

$$\pi+(1-\pi)\chi^*\alpha = \alpha\beta(1-\theta), \tag{4.8}$$

and society's welfare coincides with the case of an unconstrained economy:

$$V^I(\chi^*) = V^*, \tag{4.9}$$

[30] Without the assumption that the fee is dependent on capital income, a quadratic equation for the equilibrium value of q is obtained. Extensive numerical analysis reveals that uniqueness is attained, as one of the roots delivers a negative rental rate on capital. To streamline the analysis, I opted for the aforementioned assumption, which should be regarded as a normalization because none of the results hinges upon it. Upon request, I can provide the results for the general case in which $\chi^e = [(1-\pi)/\pi]\chi$.

[31] I denoted the quantities with a superscript I to distinguish them from the values in the previous section. The equilibrium values for the stock of capital and the wage rate can be derived from these equations.

which can be shown by directly replacing the allocations in (4.5) with χ^* in (4.6).

Equation (4.8) has a simple interpretation: dividing (4.8) by $\alpha\beta$ and multiplying by K^*:

$$\pi Y^* + (1-\pi)\chi^* r^* K^* = (1-\theta)K^*, \tag{4.10}$$

where, again, I used the Pareto allocations (3.18a) in subsection (3.3). Therefore, entrepreneurs' income πY^* is supplemented with the goods collected from lenders, the level of which was assumed to be proportional to capital income $r^* K^*$, and then, they can expand capital toward the optimal value despite the fact that θ is unchanged.

Although current lenders surrender some goods to entrepreneurs who do not consume, their constraint is relaxed for the latter and more capital is produced with general equilibrium effects that benefits all agents, including lenders whose welfare is increased. Optimal insurance can then be attained, but it requires that *status be observable*. Moreover, it would also require knowledge of θ, α and β as expressed in (4.8).

In the analysis of insurance, the imperfection that rationalizes the existence of θ is maintained. The justification was that entrepreneurs cannot sell many claims because they also act as managers of capital and could abscond with the rental income that CRS firms pay, which should be given to the actual owners of capital. Thus, even if status were observable, this moral hazard problem would justify the existence of this friction. This section assumed that the central planner was able to levy status-dependent taxes but unable to force entrepreneurs to keep their promises. Thus, in the next section, a comparison can be made between the effects of introducing money into an economy in which insurance is absent and the economy developed in this section, assuming the existence of the moral hazard problem reflected in $\theta < 1$ in both environments.

5 The economy with money

I now return to the original model with money. One can think of different equilibria that may arise with different rates of money creation γ. In each of these equilibria, agents understand that monetary policy will be maintained for the indefinite future: Money will be either injected or withdrawn at a constant rate, or the stock of money will be fixed. The natural starting point is to examine whether $\mu > 0$ when a fixed stock of money is provided in the economy.[32] The equations describing an equilibrium with constant money creation $\gamma \neq 1$ are very similar to the case in which $\gamma = 1$. Hence, to make the exposition concise, I will present the equations for the general case in which γ is not necessarily unity.

I guess that:

$$q > 1, \; \mu > 0, \tag{5.1}$$

and later find conditions such that they hold in equilibrium. Similar to the case without money, under (5.1), the feasibility set for entrepreneurs is:

$$c + q^e n' + \gamma m' \leq w + rn + m + \tau, \quad n' \geq 0, m' \geq 0, c \geq 0, \tag{5.2}$$

where q^e is defined as in (3.2). When purchasing a unit of money, the cost in terms of consumption is γ. When purchasing a unit of equity, the lenders' cost is q while the entrepreneurs' cost is q^e. To compute the expected marginal value, we need value functions. Again, I use a guess-and-verify method, assuming that the value functions are linear in states. With value functions in hand, it is also possible to find the associated policies. This is shown in Proposition 6.

Proposition 6. *Under* (5.1)*, the value and policy functions for individuals are:*

$$v(n, m; 0) = \left\{ \beta\pi \frac{q}{q^e} + 1 - \beta\pi \right\} \frac{w + \tau}{1 - \beta} + rn + m \tag{5.3a}$$

$$v(n, m; 1) = \left\{ [1 - \beta(1 - \pi)] \frac{q}{q^e} + \beta(1 - \pi) \right\} \frac{w + \tau}{1 - \beta} + \frac{q}{q^e} rn + \frac{q}{q^e} m \tag{5.3b}$$

$$g(n, m; 0) \in [0, w + rn + m + \tau - \gamma h(n, m; 0)], \quad h(n, m; 0) \in [0, w + rn + m + \tau - qg(n, m; 0)]$$

$$c(n, m; 0) = w + rn + m + \tau - qg(n, m; 0) - \gamma h(n, m; 0) \tag{5.4a}$$

$$c(n, m; 1) = 0, \quad (1 - \theta q)k'(n) = w + rn + m + \tau, \quad g(n, m; 1) = (1 - \theta)k'(n), \quad h(n, m; 1) = 0. \tag{5.4b}$$

[32] The stationary case studied here implies that regardless of the announced policy, the economy is settled in a long-run equilibrium in which all real variables are constant, including real balances. Here, we are *not* studying the effects on the transitional dynamics of introducing money into an economy with only claims on capital.

Policies in (5.4) resemble those without money. Lenders are indifferent between consuming and saving, and if they save, they are indifferent between money and claims. Entrepreneurs do not consume or purchase any money for the next period and use all resources to create capital and purchase claims. In terms of the cost of acquiring equity, entrepreneurs have an advantage because $q^e > q$, but the cost of acquiring money is the same for all and equal to γ. Values in (5.3) allow for the computation of the marginal benefits of carrying assets with which returns can be computed:

$$\underbrace{\left(\pi\frac{q}{q^e}+1-\pi\right)\frac{1}{\gamma}}_{\mathcal{M}}=\underbrace{\pi\frac{r}{q^e}+(1-\pi)\frac{r}{q}}_{\mathcal{R}_0}=\frac{1}{\beta}<\underbrace{\pi\frac{r}{q^e}\frac{q}{q^e}+(1-\pi)\frac{r}{q^e}}_{\mathcal{R}_1}. \tag{5.5}$$

In (5.5), $\mathcal{R}_0$ and $\mathcal{R}_1$ are defined as in the case without money, with the same interpretation. $\mathcal{M}$ is the return to money, which is independent of current status. The cost of purchasing one dollar is μ, and if in the next period, the agent is a lender, selling that dollar yields μ units of consumption goods if $\gamma = 1$. However, if the agent is an entrepreneur, those units of goods are not consumed but valued at rate q/q^e.[33] It is clear from (5.5) that lenders are indifferent between saving and consuming and which assets to hold. Entrepreneurs, by contrast, do not purchase money for the next period and only save in claims without consuming.

5.1 Existence of Equilibrium with Money

Lenders, being indifferent to which asset to hold, make the model undetermined at the individual level. As in the case of no money, however, any feasible amount of assets acquired by lenders yields the same individual and aggregate welfare, and aggregate quantities and prices are independent of how assets are distributed among lenders, given the existence of $\Psi(n, m)$ and its first moments. To show this, I again assume homogeneity of lenders in Assumption 2.

Assumption 2. Homogeneity of lenders' assets.

$$g(n, m; 0) = \zeta^n, \quad h(n, m; 0) = \zeta^m. \tag{5.6}$$

This assumption simply states that all lenders hold the same level of claims ζ^n and the same level of money ζ^m

[33] $1/\gamma = \mu'/\mu$, and γ being the gross rate of money creation is also equal to the gross inflation rate. Please see footnote 8 for an explanation.

independent of their current holdings; how much they hold of each is *endogenous* and given by market clearing.[34] Proposition (7) states the existence of $\Psi(n,m)$ and is shown without proof because it parallels that without money in section 3.

Proposition 7. *Under Assumption 2, a stationary distribution and density of agents with respect to assets exists, defined on the discrete support* $\left(\{n_i\}_{i=1}^{\infty}, \{m_j\}_{j=1}^{2}\right)$*:*

$$\Psi(n_i,m_j)=\begin{cases}0 & i=1;j=1\\ \pi(1-\pi^i) & i=2,3,...;j=1\\ 1-\pi^i & i=1,2,3,...;j=2,\end{cases}\qquad d\Psi(n_i,m_j)=\begin{cases}0 & i=1;j=1\\ (1-\pi)\pi^{i-1} & i=2,3,...;j=1\\ (1-\pi) & i=1;j=2\\ 0 & i=2,3,...;j=2,\end{cases}\tag{5.7}$$

where $m_1=0$, $m_2=\zeta^m$, $n_1=\zeta^n$ *and:*

$$n_i=\zeta^n\left(\frac{r}{q^e}\right)^{i-1}+\frac{w+\tau}{q^e-r}\left[1-\left(\frac{r}{q^e}\right)^{i-1}\right]+\left(\frac{r}{q^e}\right)^{i-2}\frac{\zeta^m}{q^e},\quad i=2,3,4,...\tag{5.8}$$

Note that $\int_n\int_m d\Psi(n,m)=\sum_{i=1}^{\infty}\sum_{j=1}^{2}d\Psi(n_i,m_j)=1$, as required. Finally, it is also straightforward to find the marginal densities:

$$d\Psi(n_i)=\int_m d\Psi(n,m)=(1-\pi)\pi^{i-1},\quad i=1,2,3,...;\quad d\Psi(m_j)=\int_n d\Psi(n,m)=\begin{cases}\pi & j=1\\ 1-\pi & j=2.\end{cases}\tag{5.9}$$

Proposition 7 characterizes the distribution of agents with respect to assets, which has an special structure. At this point, the existence of equilibrium remains to be shown. However, for the sake of exposition, let me explain how transactions are conducted and how money circulates in the economy. To this end, I present in Figure 1 a numerical example of the density that emerges in (5.7).[35]

[34] Similar considerations regarding the role of full depreciation, as in the case of no money in Section 3, and Assumption 1 are valid in the present case.

[35] Subsequently, we will see that $r>q^e$ in equilibrium and hence the support of the distribution is unbounded in the dimension of equity. The figure, hence, shows only a portion of this density.

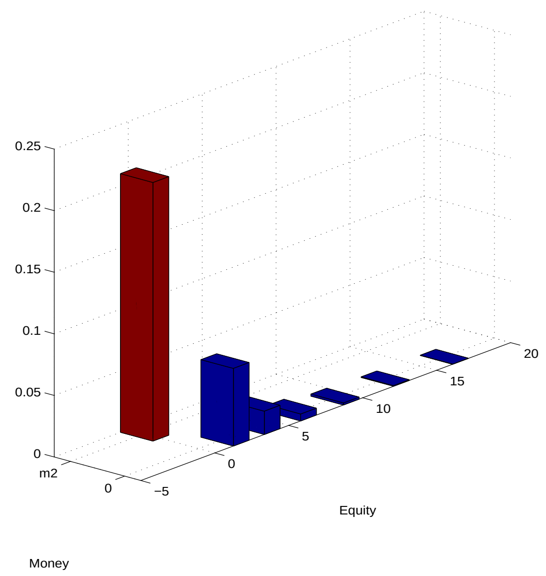

Figure 1: $d\Psi(n, m)$:
The graphic shows a portion of the density of agents by assets. The values used are: π=0.3, α=0.36, β=0.95, θ=0.01 and γ=1. The resulting value for ζ^n is 0.021, which is indistinguishable from zero in the figure.

Two financial transactions are conducted at each moment in time: trades of equity for goods and trades of money for goods. $1-\pi$ individuals is the density in state $(n_1, m_2) = (\zeta^n, \zeta^m)$, and in states $(n_i, m_1)_{i=2}^{\infty}$, there are $(1-\pi)\pi^{i-1}$ individuals, which is expressed in (5.7). At each of these points, the mass of agents is divided between π entrepreneurs and $1-\pi$ lenders. How do individuals move in this economy? Of the $1-\pi$ individuals in (n_1, m_2), π become entrepreneurs; these agents sell money and go to point (n_2, m_1). A fraction π of all agents in $(n_i, m_1)_{i=2}^{\infty}$, each of measure $(1-\pi)\pi^i$, remain entrepreneurs, which means that they move to (n_{i+1}, m_1), $i \geq 2$. Similarly, of $1-\pi$ individuals in (n_1, m_2), $1-\pi$ remain lenders and hence remain at that point. A fraction $1-\pi$ of all agents in $(n_i, m_1)_{i=2}^{\infty}$, each of measure $(1-\pi)\pi^i$, become lenders, and these go to point (n_1, m_2).

Note that there is a fraction of the stock of money in each period that remains "idle" in the sense that it is not transacted. It is held by lenders waiting to become entrepreneurs; once they find an investment opportunity, their money will serve as "start-up" capital because in state (n_1, m_2), equity holdings are lowest. For this reason, demand for money is precautionary.

To find the aggregate equilibrium of the economy, I once again employ a guess-and-verify method. I assume that $\int n d\Psi(n) < \infty$, which is not obvious given the infinite structure in (5.8), and then verify the condition.

The relevant equations are indifference in returns for lenders expressed in (5.5), the aggregate version of capital accumulation of entrepreneurs, from (5.4b):

$$\underbrace{(1-q\theta)K}_{\text{down payment}} = \underbrace{[w+\gamma H+rK]\,\pi}_{\text{entrepreneurs' own funds}}, \tag{5.10}$$

and equilibrium rental factor prices in (3.13c). In (5.10), it is clear that to finance an investment down payment, all factor income is used in addition to all real money holdings in entrepreneurs' hands.[36] The equations form a system of equations with solutions:[37]

$$q^M = \frac{\beta\pi+\gamma-\beta}{\beta\pi+(\gamma-\beta)\theta}, r^M = \frac{\beta\pi+\gamma-\beta}{[\beta\pi+(\gamma-\beta)\theta]\,\gamma}, K^M = \left(\frac{\alpha}{r^M}\right)^{\frac{1}{1-\alpha}}, H = \frac{(1-\theta)\alpha\beta\gamma-\beta\pi-\gamma+\beta}{\alpha\gamma^2\,[\beta\pi+(\gamma-\beta)\theta]}K^M. \tag{5.11}$$

It was conjectured that $q > 1$ in equilibrium, as (5.11) shows, $q^M > 1$ if $\theta < 1$, which is true by assumption. The condition for money to be valued $\mu > 1$ is then found using aggregate real balances in (5.11) because for aggregate real balances to be strictly positive, and hence for money to be valued in equilibrium, it must be the case that:

$$\pi + \frac{\gamma-\beta}{\gamma}(1-\pi) < \alpha\beta(1-\theta). \tag{5.12}$$

In the next subsection, I explain the significance of inequality (5.12). Now, I complete the description of the existence of equilibrium with the remaining details. As mentioned above, average quantities of the form $\int m d\Psi(m)$ and $\int n d\Psi(n)$ were used throughout. While the former is guaranteed to hold, the later is not obvious. The following proposition establishes that while the support on equity is not bounded above, average equity is well defined in the monetary economy.

Proposition 8. *$\mathcal{B}$ is not bounded above in the dimension of equity and:*

$$\int n d\Psi(n) < +\infty. \tag{5.13}$$

[36]Note that in aggregating the capital accumulation equation in (5.4b), money-related terms satisfy: $\pi\int[m+\tau]d\Psi(m) = \pi[H+\mu T] = \pi[H+\mu(\gamma-1)M] = \pi\gamma H$.

[37]With these closed-form solutions, it is possible to examine the effects of changing parameters θ and π, as was done in 3.17 for the case of no money. Similar effects are found, and I omit this discussion here. What is new here is the effects of changing γ and its relationship with β; this is examined in the next section.

Finally, market clearing provides the endogenously determined values of ζ^n and ζ^m, which satisfy:

$$(1-\pi)\zeta^n = \theta K^M, \quad (1-\pi)\zeta^m = H, \tag{5.14}$$

where the equilibrium values of K^M and H are given in (5.11). These conditions state that aggregate claims in lenders' hands are equal to the fraction of capital on which claims are issued and that the aggregate real stock of money is accepted by lenders. Let me elaborate on this last point by making a reference to the distribution in (5.7). Note that the π becoming entrepreneurs from point (n_1, m_2) sell $\pi\zeta^m(1-\pi)$ money to lenders. However, lenders who remain lenders at that point are still holding $(1-\pi)\zeta^m(1-\pi)$ units of money. Of course, this stock of money is preserved in the system: $(1-\pi)\zeta^m(1-\pi)+\pi\zeta^m(1-\pi) = \zeta^m(1-\pi) = H$. However, who is acquiring the $\pi\zeta^m(1-\pi)$ units of money? It is those agents becoming lenders, who, by examining (5.7), are given by: $(1-\pi)\sum_{j=2}^{\infty}(1-\pi)\pi^{i-1} = (1-\pi)\pi$. Each of them holds ζ^m units of money.

When becoming entrepreneurs, agents going from (n_1, m_2) to (n_2, m_1), deplete a given stock of money in their hands, the amount of goods obtained with money along with factor income are used to purchase equity. Income from equity in the next period will be used again to purchase more equity if entrepreneurs repeatedly find investment opportunities. Therefore, the influence of money on funding investment will persist over time, which is why the term ζ^m appears in (5.8) for an arbitrary i. This feature of the model differs from constructs in the "search" tradition of models with money, as in Nosal and Rocheteau (2013) and Telyukova and Visschers (2013). In such models, there is a division of each period into "centralized" and "decentralized" subperiods. This feature allows one to consider heterogeneity that matters only between subperiods, but the distribution is reset for the subsequent period, and hence, any choice of assets matters only for the adjacent subperiod.[38]

In this environment, an important question is whether, by providing more money, the monetary authority actually increases τ in (5.8) and thereby induces more capital creation. Perhaps a more basic question is whether by simply introducing a stock of money into the economy, social welfare is improved. These questions are examined in the next subsection after completing the description of the monetary equilibrium, which includes an explanation of the significance of condition (5.12). Moreover, I discuss whether money improves the economy. For this purpose,

[38] I thank an anonymous referee for bringing attention to how papers in the "search" tradition of money address heterogeneity and its limitations and how the present paper differs from and improves on them in some dimensions.

social welfare can be computed with (5.3) and (5.11) in closed form using:[39]

$$V^M = (1-\pi)\int v(n,m;0)d\Psi(n,m) + \pi\int v(n,m;1)d\Psi(n,m). \tag{5.15}$$

5.2 Efficiency Properties of the Equilibrium with Money

Money is valued when (5.12) is satisfied. Dividing (5.12) by $\alpha\beta$ and multiplying by K^*, under $\gamma = 1$, yields:

$$\pi Y^* + (1-\beta)(1-\pi)Y^* < (1-\theta)K^*. \tag{5.16}$$

Condition (5.16) reveals that money is valued when, despite entrepreneurs devoting their entire income plus being supplemented with a fraction $1-\beta$ of lenders' income, entrepreneurs do not obtain sufficient resources to self-finance the optimal stock K^*. In analyzing the model without money, Inequality (3.16) yielded a condition such that $q > 1$. When comparing that condition with (5.12), it follows that for money to be valued, the economy needs to be more constrained, either on the extensive margin with fewer entrepreneurs (lower π) or with a tighter finance constraint (lower θ). To understand this result, note that unlike equity, money does not pay any return on itself. If the constraint parameters θ and π are not sufficiently small such that q/q^e is not high enough, then $\mathcal{M}$ in (5.5) is lower than $1/\beta$, and hence, lenders would not be willing to hold money and money would not be valued.

Conditions (3.16) and (5.12) under $\gamma = 1$ can be used to portray combinations of π and θ that define whether there is inefficiency in the economy and whether money is valued; this can be seen in Figure 2.

[39]A closed-form solution for this value function is presented in Equation (A.10) in the Appendix, in the proof of Proposition 9.

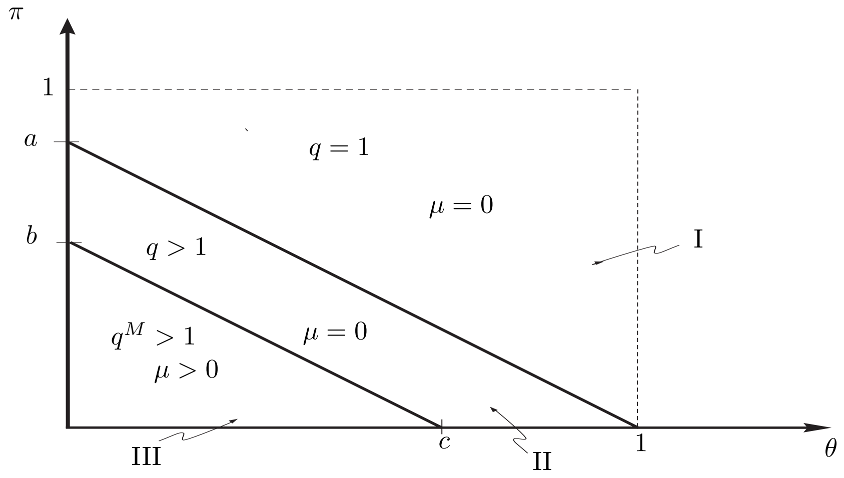

Figure 2: Regions of Efficiency:
With a given stock of money $\gamma = 1$; when the economy is relatively unconstrained (sufficiently high values of θ and π), the economy attains a first-best equilibrium; there is no role for money. When the economy is constrained, money may be valued. $a = \alpha\beta$, $b = \frac{\alpha\beta-(1-\beta)}{\beta}$, and $c = \frac{\alpha\beta-(1-\beta)}{\alpha\beta}$

In regions I and II, the comparison made in the previous section between the economy with and without money is valid; in region II, condition (3.21) is satisfied. What happens in region III? In this region, money is valued, and credit or equity and money coexist in this region. This model thus resembles some features of models developed in the "monetary search" constructs, such as those of Aiyagari, Wallace and Wright (1996), Mills (2007), Telyukova and Wright (2008) and, especially, Nosal and Rocheteau (2013). As credit helps to attain better outcomes and money is valued, can it be the case that $V^M = V^*$? The next proposition establishes that while welfare is improved relative to the absence of money, the same welfare as that in a frictionless economy is not attained.

Proposition 9. *Under Condition* (5.12) *(when* $\gamma = 1$*):*

$$V < V^M < V^*. \tag{5.17}$$

Thus, while money improves welfare, it is not alone able to attain optimality. Active monetary policy may be needed to accomplish this, which is investigated in the next section.

6 Non-neutrality and the Optimality of the Friedman Rule

Imagine that an economy is initially in a steady state with a constant stock of money such that it is valued. I wish to assess the effects of different rates of money creation. Analyzing the transition from an equilibrium with a constant stock of money to another with a constant rate of growth is beyond the scope of this paper. I simply assume that once γ is set at a value other than one, the economy settles again into a stationary situation, and then, I explore the effects of money injections or subtractions. By assumption, money is injected into the economy as a lump sum and proportionally to all agents. Furthermore, if money is taxed away, this proportionally affects all agents in the economy; the monetary authority is unable to identify the status of each agent and conduct targeted monetary injections or subtractions.

The next proposition characterizes the different equilibria and their allocation and welfare properties as a function of γ.

Proposition 10. *Assume that the economy is sufficiently constrained such that* (5.12) *holds; there is a non-empty set* $[\beta, \bar{\gamma}]$ *to which* γ *belongs and will induce the following properties in the model:*

1. *For* $\gamma \in (\beta, \bar{\gamma})$*:*

$$\frac{\partial q^M}{\partial \gamma} > 0, \quad \frac{\partial K^M}{\partial \gamma} < 0, \quad \frac{\partial H}{\partial \gamma} < 0, \quad \frac{\partial (\gamma H)}{\partial \gamma} < 0, \quad \frac{\partial V^M}{\partial \gamma} < 0. \tag{6.1}$$

2. *Money may cease to be valued: for* $\gamma = \bar{\gamma}$, $V^* > V^M = V$, $1 < q^M = q$ *and* $\mu = 0$.[40]

3. *Optimality of the Friedman rule: if* $\gamma = \beta$*, then* $q^M = 1$ *and* $V^M = V^*$.

Figure 3 illustrates the characterization of Proposition 10.

[40] If $\gamma > \bar{\gamma}$, then money is dropped altogether and q^M and V^M are not defined.

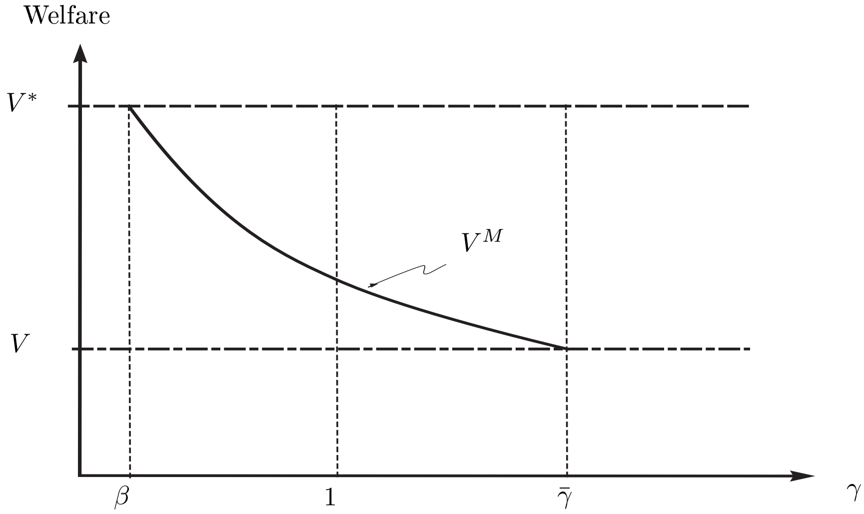

Figure 3: Welfare as a function of γ:
Ex ante welfare as a function of γ: $\gamma = \beta$ attains optimality; as γ increases, welfare decreases until $\gamma = \bar{\gamma}$, where inflation is so high that money ceases to be valued.

In general, the higher the inflation rate is, the more detrimental its effect on the economy. The return to money $\mathcal{M}$, defined in (5.5), declines with inflation; this is straightforward and is a result in most flexible price models. However, in this linear-utility model, a decrease in the return to money implies that there would be zero demand for real balances because lenders have other assets to save. A decrease in the demand for money would depend on the price of money decreasing to induce lenders to demand money again. However, in this setup, a decrease in the price of money has no effect on the return on money per se. There are general equilibrium effects that involve the investment financing channel that influences q. In Equation (5.10), the term $\gamma H\pi = \gamma\mu M\pi$ equals the fraction of real balances devoted to financing investment down payments. These balances include the lump-sum transfer of money by the monetary authority. In Equations (6.1), we can see that not only do real balances H decrease with money injections, but the sum of real balances plus injections of money γH do as well. This reduces entrepreneurs' resources, and it follows that the price of money is decreasing significantly. The reason, as stated above, is that lenders are unwilling to demand any money in the market. The decrease in the price of money tightens the entrepreneurs' constraint to the extent that capital shrinks and claims on it become more valuable. This drives up the ratio q/q^e, which in turn increases $\mathcal{M}$ enough to restore the return on money to the discount rate $1/\beta$ in (5.5). Thus, an economy with higher rates of money creation yields a lower aggregate capital stock and a higher ratio q/q^e. As discussed previously, this also means that there is more divergence in the idiosyncratic marginal rates of transformation, which is welfare-detrimental for individuals.

At relatively low levels of inflation, welfare is higher than in the absence of money. In particular, just a single stock of money is valuable and welfare-enhancing. Nevertheless, if the growth rate of money is high enough, when $\gamma = \bar{\gamma}$, the equilibrium allocations and welfare are exactly what would have been obtained in the absence of money and money is no longer valued.

Deflation is welfare-enhancing; although there are perpetual withdrawals of money in this case, actual real balances for financing capital creation are higher. Deflation increases the return on money, and lenders demand more of it. The resulting increase in the price of money allows more transfers of resources from lenders to entrepreneurs, precisely the desirable effect that insurance would have if it were feasible. The relaxation of the entrepreneurs' constraint decreases q and the ratio q/q^e, which in turn decrease the return $\mathcal{M}$ to $1/\beta$. Again, the decrease in the ratio q/q^e means that the marginal rates of transformation among individuals are closer.

A deflationary policy in which $\gamma = \beta$ attains optimal allocations and the same welfare as perfect insurance. For example, from (5.11): $q^M = 1, K^M = K^*$ and $r^M = 1/\beta = r^*$. The liquidity needs of entrepreneurs are satiated because γH is high enough that the optimal capital stock K^* can be financed. Note from (5.5) that in this case:

$$\mathcal{M} = \mathcal{R}_0 = \frac{1}{\beta} = r^* = \mathcal{R}_1. \tag{6.2}$$

The model is indeterminate at the individual level, but this is utility-irrelevant.[41] The result obtained is the same as relaxing θ until the economy is unconstrained, and we can still assume that entrepreneurs consume zero and undertake investment. Because, in this situation, $q^M = 1$, there is no difference in the marginal rates of transformation among individuals. When $\gamma > \beta$, an entrepreneur finds money to be dominated in return and optimally does not hold any for the next period. In effect, money is an inferior asset with respect to equity, which has an effective expected return of $\mathcal{R}_1 > \mathcal{M}$, as stated in (5.5). The Friedman rule eliminates such an inferiority by equating all asset returns to $1/\beta$. As stated in the introduction, some authors note that the Friedman rule attains optimality in versions of the Kiyotaki and Moore setup. For example, Kocherlakota (2005) states that the Friedman rule would attain optimality in Kiyotaki and Moore (2005); to my knowledge, the exposition in

[41] Nosal and Rocheteau (2013) also find that by implementing the Friedman rule, the monetary authority induces the returns on the real asset and money to be identical and equal to the rate of time preference. However, they find that if $\gamma > \beta$, then there is a difference in the return on those assets. In the model used in this paper, returns are always the same regardless of the inflation level as long as money is valued. Linearity of preferences is responsible for this result, as there is no liquidity premium for money.

this paper is the first to formalize the result.

Note that even a slight increase in deflation β is detrimental to welfare. Nosal and Rocheteau (2013) find that there is a range of inflation rates above β that sustain optimal allocations. The implications of this finding are important because it implies that low inflation rates do not entail welfare costs. Their model emphasizes a notion of liquidity defined over a real asset in *fixed* supply. Thus, while the economy may present liquidity shortages, low levels of inflation that increase the cost of holding real balances do not have a first-order impact on the first-best amount of goods that can be created in the economy, which is independent of the fixed asset. In the model developed in this paper, the real asset (the capital stock) is *endogenous* and sensitive to the liquidity properties of claims, as expressed in $\theta < 1$. Then, even low levels of inflation have first-order effects on the resources available to entrepreneurs: their real balances, w and rK, which depend on the capital stock. This implies that results regarding the innocuousness of low inflation in Nosal and Rocheteau (2013) may not be robust once endogeneity of the real asset is taken into account.[42]

If the money growth rate decreases below β, then money is so profitable that there is no demand for equity, nor production of new capital, which cannot be an equilibrium.

Finally, I discuss the existence of the distribution and its first moment under the Friedman rule. Note that real balances are well defined, as when $\gamma = \beta$, (5.11) implies that $H = [(1-\theta)\alpha\beta - \pi]/(\alpha\beta^2\pi)$. Thus, Condition (3.16) suffices for $H > 0$. Recall that (3.16) imposed conditions on the parameters such that $q > 1$ and the economy attained suboptimal allocations, where money was absent. Average holdings are also well defined because the boundedness of $\int n d\Psi(n)$ requires from (5.8) that $\pi < \beta$, but this follows again from (3.16) because $\pi < \alpha\beta(1-\theta) < \beta$. Hence, a constrained economy in the absence of money is sufficient to guarantee that a monetary equilibrium exists under the Friedman rule.

In the analysis above, θ was held fixed. The model is vulnerable to a "Lucas critique" because monetary policy may affect incentives that underlie the moral hazard problem reflected in θ. Recall that entrepreneurs also act as managers of capital: They build capital goods and rent them to the CRS firms, giving the rental income to the actual owners of capital. If implementing the Friedman rule yields an increase in the price of money and

[42]In Nosal and Rocheteau (2013), even if $\theta = 0$, q can attain its "efficient" fundamental value and allocations may be optimal. Moreover, q may be higher than its fundamental value and the allocations may still be optimal. In the model developed in this paper, there is a one-to-one mapping of q and welfare. In both cases, with or without money, we can see that whenever $\theta = 0$, q takes its maximum value and K its minimum value, for given parameters; see (3.14) and (5.11). Then, it is not possible to uncouple liquidity from capital creation, which, by assumption, is possible in Nosal and Rocheteau (2013).

relaxes the entrepreneurs' constraint, then the extra capital created may aggravate the moral hazard problem, as the temptation to abscond with the proceeds of capital is higher. Knowing this, lenders may further decrease the supply of funds to entrepreneurs, which would counteract the beneficial effect of the deflationary policy. However, a better assessment of the potential effect of monetary policy on the moral hazard problem requires a model that explicitly addresses the informational problem; this is left for future work.

7 Conclusion

The model developed in this paper contributes to the understanding of the effects of financial constraints on allocations and welfare in general equilibrium models with unobservable idiosyncratic investment opportunities and the role of money and monetary policy. I show how transactions of claims on capital are used as self-insurance for the event of finding investment opportunities and upon which individuals desire to have ample funds to take maximum advantage of such an opportunity. Financial constraints prevent a sufficient flow of resources among individuals to fully finance the investment projects, and hence, they prevent the economy from achieving maximal welfare. I show how a social planner that is able to identify who has an investment opportunity could implement efficient outcomes and maximal welfare and how the same efficient outcomes and welfare can be attained by implementing the Friedman rule without any requirement of the observability of an individual's status regarding having an investment opportunity.

As for possible implications of the theory presented here, recall that lenders hold money in this model to partially overcome the lack of credit in terms of equity provided by entrepreneurs. Hence, an economy in which individuals hold or accumulate substantial amounts of money would be interpreted, through the lens of this model, as an economy in which informational problems disrupt efficient financial transactions. Perhaps when some agents hold large amounts of money, as the corporate sector did in the U.S. economy after the 2008 financial crisis, this is symptomatic of special periods of financial market distress due to the increased difficulty encountered by private institutions in eliciting accurate information from market participants.

A Appendix

Proposition 1.

Proof. With conjectures in (3.4), the Bellman equations are:

$$
\begin{aligned}
v(n;0) &= \max_{0\leq n'\leq \frac{w+rn}{q}} \left[w + rn - qn' + \beta\bar{A} + \beta\bar{B}n'\right] && \text{(A.1a)}\\
v(n;1) &= \max_{0\leq n'\leq \frac{w+rn}{q^e}} \left[w + rn - q^e n' + \beta\bar{A} + \beta\bar{B}n'\right], && \text{(A.1b)}
\end{aligned}
$$

for entrepreneurs and lenders, respectively, where $\bar{A} = \pi A_1 + (1-\pi)A_0$ and $\bar{B} = \pi B_1 + (1-\pi)B_0$.

The margins that matter are the marginal gain from acquiring equity $\beta\bar{B}$ and the marginal cost q and q^e for lenders and entrepreneurs, respectively. In principle, five cases may arise: i) $\beta\bar{B} < q^e < q$, ii) $q^e = \beta\bar{B} < q$, iii) $q^e < \beta\bar{B} < q$, iv) $q^e < q = \beta\bar{B}$ and v) $q^e < q < \beta\bar{B}$.

Note that under $0 < \theta < 1$, cases i) through iii) cannot arise in equilibrium because no lender will be willing to purchase any claims. In case i), furthermore, entrepreneurs are not motivated to create any capital. Case v) can also be excluded because no agents would ever consume, and hence, it cannot be an equilibrium. The only possible equilibrium entails case iv), from which policies in (3.6) are deduced. With the standard method of equating coefficients, given policies and using the fact that $\beta\bar{B} = q$, it is straightforward to obtain value functions in (3.5). □

Proposition 2.

Proof. This proof consists of several steps. First, I show that the support of the stationary distribution is countably infinite.

Step 1: $\Psi(n)$ has a discrete, countably infinite support.

To show step 1, assume that under fixed prices, individuals are "initialized" arbitrarily along $[0, +\infty)$ in $\mathcal{B}$; given Assumption 1, each individual will eventually reach ζ and remain there as long as he is a lender. If he becomes an entrepreneur, then the policy for equity in (3.6b) applies. It follows that all agents will hold equity only in the states defined by the following recursion:

$$n_{i+1} = \frac{w + rn_i}{q^e}, \tag{A.2}$$

with initial condition $n_1 = \zeta$. This difference equation has a unique solution that is precisely the support of the distribution in (3.12).

Step 2: The law of motion for the distribution of agents with respect to equity follows:

$$\Psi\left(n_{i+1}\right) = \pi\Psi\left(n_i\right) + 1 - \pi, \tag{A.3}$$

where n_i is defined in (3.12).

By (2.7f), stationarity of the measure of agents requires:

$$\Psi(n') = \pi \int_{n:n\geq\zeta, g(n;1)\leq n'} d\Psi(n) + (1-\pi)\int_{n:n\geq\zeta, g(n;0)\leq n'} d\Psi(n) = \pi\Psi\left(\frac{q^e n' - w}{r}\right) + (1-\pi). \tag{A.4}$$

Because the support in (3.12) is discrete, the measure is zero except at the discrete points, where the measure follows (A.4):

$$\Psi\left(n_{i+1}\right) = \pi\Psi\left(\frac{q^e n_{i+1} - w}{r}\right) + (1-\pi) = \pi\Psi\left(n_i\right) + (1-\pi).$$

This is a first-order difference equation with boundary initial condition: $\Psi(n_1) = 1 - \pi$. The solution for this equation is given by (3.11). □

Proposition 3.

Proof. First, I show that the support of $\Psi(n)$ is unbounded above. In (3.12), an unbounded support means that $r/q^e \geq 1$. Directly from (3.14), it is possible to find:

$$\frac{r}{q^e} = \frac{\alpha(1-\theta)}{\pi}. \tag{A.5}$$

By way of contradiction, assume that $\alpha(1-\theta) < \pi$, then

$$\alpha\beta(1-\theta) < \alpha(1-\theta) < \pi. \tag{A.6}$$

However, this violates (3.16), and hence, $\alpha(1-\theta) \geq \pi$. To show that despite the unboundedness of the support of the distribution, the mean $\int n d\Psi(n)$ is well defined, take (3.11) and (3.12):

$$\int_{\zeta}^{\infty} n d\Psi(n) = \sum_{i=1}^{\infty} n_i \Psi_i = \sum_{i=1}^{\infty} \left\{ \zeta \left(\frac{r}{q^e}\right)^{i-1} + \frac{w}{q^e - r}\left[1 - \left(\frac{r}{q^e}\right)^{i-1}\right]\right\} (1-\pi)\pi^{i-1}.$$

The infinite summations in this expression will converge if and only if $\frac{r\pi}{q^e} < 1$. However, from (A.5), this condition is satisfied directly because $\alpha(1-\theta) < 1$. □

Proposition 4

Proof. With prices and allocations in (3.14), is possible to find a closed-form solution for welfare in (3.9):

$$V = \frac{1-\alpha\beta}{\alpha(1-\beta)}\left(1 - \pi + \pi\frac{q}{q^e}\right) rK = \frac{(1-\pi)(1-\alpha\beta)}{[1-\alpha\beta(1-\theta)](1-\beta)}\left[\frac{\pi\left(\frac{1}{\alpha\beta} - 1\right) + \theta}{\frac{1}{\alpha\beta} - 1 + \theta}\right]^{\frac{\alpha}{1-\alpha}}. \tag{A.7}$$

By way of contradiction, assuming that (3.16) holds and $V^* < V$, then:

$$\frac{V}{V^*} > 1 \rightarrow (1-\pi)\left[\pi\left(\frac{1}{\alpha\beta} - 1\right) + \theta\right]^{\frac{\alpha}{1-\alpha}} > [1-\alpha\beta(1-\theta)]^{\frac{1}{1-\alpha}}$$

using (3.18b) and (A.7). Note that if (3.16) holds, then $[1-\alpha\beta(1-\theta)]^{\frac{1}{1-\alpha}} > [1-\pi]^{\frac{1}{1-\alpha}}$, and hence, the inequality above implies:

$$(1-\pi)\left[\pi\left(\frac{1}{\alpha\beta} - 1\right) + \theta\right]^{\frac{\alpha}{1-\alpha}} > [1-\pi]^{\frac{1}{1-\alpha}}.$$

This inequality would hold when $\pi\left(\frac{1}{\alpha\beta} - 1\right) + \theta > (1-\pi)$, but this inequality is $\pi > \alpha\beta(1-\theta)$, violating (3.16). □

Proposition 6.

Proof. I conjecture that the value functions are linear in the states n and m.

$$v(n,m;z) \quad = \quad A_z + B_z n + C_z m, \quad z = \{0,1\},$$

where A_z, B_z and C_z are coefficients to be determined. The Bellman equation for the entrepreneur is:

$$v(n,m;1) = \max_{c,n',m'} \left[c + \beta\bar{A} + \beta\bar{B}n' + \beta\bar{C}m'\right]$$

subject to:

$$c + q^e n' + \gamma m' \leq w + \tau + m + rn, \quad c \geq 0, \quad n' \geq 0, \quad m' \geq 0,$$

and that for the lender is:

$$v(n,m;0) = \max_{c,n',m'} \left[c + \beta\bar{A} + \beta\bar{B}n' + \beta\bar{C}m'\right]$$

subject to:

$$c + qn' + \gamma m' \leq w + \tau + m + rn, \quad c \geq 0, \quad n' \geq 0, \quad m' \geq 0,$$

where: $\bar{A} = \pi A_1 + (1-\pi)A_0, \bar{B} = \pi B_1 + (1-\pi)B_0$ and $\bar{C} = \pi C_1 + (1-\pi)C_0$.

In this environment, what matters is the returns on the different assets. Note that an equilibrium for the equity market can only arise under $q = \beta\bar{B}$, and thus, in principle, five different possibilities may arise: i) $\frac{\bar{C}}{\gamma} < \frac{1}{\beta} = \frac{\bar{B}}{q} < \frac{\bar{B}}{q^e}$, ii) $\frac{\bar{C}}{\gamma} = \frac{1}{\beta} = \frac{\bar{B}}{q} < \frac{\bar{B}}{q^e}$, iii) $\frac{1}{\beta} = \frac{\bar{B}}{q} < \frac{\bar{C}}{\gamma} < \frac{\bar{B}}{q^e}$, iv) $\frac{1}{\beta} = \frac{\bar{B}}{q} < \frac{\bar{B}}{q^e} = \frac{\bar{C}}{\gamma}$ and v) $\frac{1}{\beta} = \frac{\bar{B}}{q} < \frac{\bar{B}}{q^e} < \frac{\bar{C}}{\gamma}$.

Case i) cannot be excluded because money in this model is not forced to fulfill a specific function. Case ii) is also possible; in this case, entrepreneurs will sell their money holdings, as money for them is dominated with respect to return. Lenders are indifferent between holding money or equity as a means of saving. Cases iii) through v) can all be excluded because money would either dominate the return on equity for lenders, entrepreneurs or both, and then, there would be no market for equity and no investment would be undertaken. When case ii) holds, policy functions in (5.4) can be deduced.

With policy functions so defined, is possible to use policies in (5.4) and the standard method of equating coefficients of the value functions; this delivers (5.3). □

Proposition 8.

Proof. [43] To show that the support is in fact unbounded above along equity, it suffices to show that $r/q^e > 1$. However, closed-form solutions are readily available, and I find that this inequality holds as long as: $\gamma - \beta + \beta\gamma(1-\pi) > 0$. As γ is restricted to being no less than β, the result follows. Now, note that $\int nd\Psi(n)$ is:

$$\int_{\zeta^n}^{\infty} nd\Psi(n) = \sum_{j=1}^{\infty} n_i\Psi_i = \sum_{i=1}^{\infty}\left\{\zeta^n\left(\frac{r}{q^e}\right)^{i-1} + \frac{w+\tau}{q^e-r}\left[1-\left(\frac{r}{q^e}\right)^{i-1}\right] + \left(\frac{r}{q^e}\right)^{i-2}\frac{\zeta^m}{q^e}\right\}(1-\pi)\pi^{i-1}. \quad \text{(A.8)}$$

Clearly, the infinite summations will converge if and only if $\frac{r\pi}{q^e} < 1$. Directly using the closed-form solutions for prices, this requires:

$$\frac{r\pi}{q^e} = \frac{\gamma - \beta(1-\pi)}{\gamma\beta} < 1. \quad \text{(A.9)}$$

By way of contradiction, assume that: $\gamma - \beta(1-\pi) \geq \gamma\beta$. By manipulating this expression, I find: $\pi + \frac{\gamma-\beta}{\gamma}(1-\pi) \geq \beta$. Then, the following inequality would be satisfied:

$$\pi + \frac{\gamma-\beta}{\gamma}(1-\pi) \geq \beta > \alpha\beta(1-\theta).$$

However, this violates (5.12). □

Proposition 9.

Proof. With the prices and allocations in (5.11), is possible to find a closed-form solution for welfare in (5.15):

$$V^M = \left(1 - \pi + \pi\frac{q^M}{q^{eM}}\right)\left(\frac{w^M}{1-\beta} + \frac{\gamma-\beta}{1-\beta}H + r^M K^M\right) = \frac{\pi + \frac{\gamma-\beta}{\beta} - \alpha\gamma\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)}{(1-\beta)\left(\pi + \frac{\gamma-\beta}{\beta}\right)}\left[\frac{\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)\alpha\gamma}{\pi + \frac{\gamma-\beta}{\beta}}\right]^{\frac{\alpha}{1-\alpha}} \quad \text{(A.10)}$$

By way of contradiction, assume that (5.12) holds and $V > V^M$, then:

$$\frac{V}{V^M} > 1 \rightarrow \frac{(1-\pi)(1-\alpha\beta)}{-\beta\left[\pi + \frac{\gamma-\beta}{\beta} - \alpha\gamma\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)\right]}\left(\frac{\pi(1-\alpha) + \theta\alpha\beta}{\alpha\gamma[\pi\beta + \theta(\gamma-\beta)]}\right)^{\frac{\alpha}{1-\alpha}} > \left(\frac{1-\alpha\beta(1-\theta)}{\pi\beta + \gamma - \beta}\right)^{\frac{1}{1-\alpha}}.$$

Given the negative sign in the denominator of the LHS of this inequality, the only possibility for this inequality to hold is when the term in brackets in the first ratio is negative, but this would imply that $\beta\pi(1-\alpha\gamma) + (\gamma - \beta)(1-\alpha\gamma\theta) < 0$, which is not true when $\gamma = 1$. To show that $V^M < V^*$, again by way of contradiction assuming

[43] This proposition will be shown for all $\gamma \geq \beta$, which is also useful for Section 6. Of course, results will hold in particular for $\gamma = 1$.

that $V^M > V^*$:

$$\frac{V^M}{V^*} > 1 \to \frac{\pi + \frac{\gamma-\beta}{\beta} - \alpha\gamma\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)}{1-\alpha\beta}\left[\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)\frac{\gamma}{\beta}\right]^{\frac{\alpha}{1-\alpha}} > \left(\pi + \frac{\gamma-\beta}{\beta}\right)^{\frac{1}{1-\alpha}}. \tag{A.11}$$

Now assuming that:

$$\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)\frac{\gamma}{\beta} > \frac{\pi + \frac{\gamma-\beta}{\beta} - \alpha\gamma\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right)}{1-\alpha\beta}, \tag{A.12}$$

inequality (A.11) implies:

$$\left(\pi + \frac{\gamma-\beta}{\beta}\right)(1-\alpha\beta) > \pi + \frac{\gamma-\beta}{\beta} - \alpha\gamma\left(\pi + \frac{\gamma-\beta}{\beta}\theta\right). \tag{A.13}$$

When $\gamma = 1$, both this last inequality and (A.12) will be satisfied when $\pi > 1-\theta$, but this means that $\pi > \alpha\beta(1-\theta)$, violating (3.16). □

Proposition 10.

Proof. 1. These results follow from direct differentiation of the equations in (5.11) and (A.10).

2. $\bar{\gamma}$ is defined as the value of γ such that $q = q^M$. After some algebra, I obtain:

$$\bar{\gamma} = \frac{(1-\pi)\beta}{1-\alpha\beta+\alpha\beta\theta}. \tag{A.14}$$

To show that $\bar{\gamma} > \beta$, and hence the set is non-empty, by way of contradiction, assuming: $\beta(1-\pi) \leq \beta[1-\alpha\beta+\alpha\beta\theta]$, this implies:

$$\pi \geq \alpha\beta(1-\theta),$$

violating (3.16).

To show that this value would also make money valueless, replacing $\bar{\gamma}$ in the right-hand side of Inequality (5.12):

$$\pi + \frac{\bar{\gamma}-\beta}{\beta}(1-\pi) = \alpha\beta(1-\theta), \tag{A.15}$$

violating (5.12).

3. Follows directly by setting $\gamma = \beta$ in the prices and allocations in (5.11) and in (A.10)

$\square$

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